

Tensor Part-VII

Quotient Law:

Consider ~~a~~ a quantity A and we do not know whether A is tensor or not. If the inner product of A with an arbitrary tensor is itself a tensor then A is also a tensor. This is called the quotient law.

Let $A(i, j, k) \rightarrow$ components of a tensor

Let inner product of ~~A~~ $A(i, j, k)$ with an arbitrary tensor B^{pq} is a contravariant tensor of first rank

$$A(i, j, k) B^{jk} = C^i \quad \text{--- (1)}$$

j, k (Repeated index) are summed over.

Let $\bar{A}(\alpha, \beta, \gamma) \rightarrow N^3$ functions in the barred coordinates and satisfy

$$\bar{A}(\alpha, \beta, \gamma) \bar{B}^{\beta\gamma} = \bar{C}^\alpha \quad \text{--- (2)}$$

to write \bar{B} ~~and~~ and \bar{C} in terms of unbarred components, we write the above eq. as

$$\bar{A}(\alpha, \beta, \gamma) \frac{\partial \bar{x}^\beta}{\partial x^j} \frac{\partial \bar{x}^\gamma}{\partial x^k} B^{jk} = \frac{\partial \bar{x}^\alpha}{\partial x^i} C^i$$

$$= \frac{\partial \bar{x}^\alpha}{\partial x^i} A(i, j, k) B^{jk}$$

or $\left[\bar{A}(\alpha, \beta, \gamma) \frac{\partial \bar{x}^\beta}{\partial x^j} \frac{\partial \bar{x}^\gamma}{\partial x^k} - \frac{\partial \bar{x}^\alpha}{\partial x^i} A(i, j, k) \right] B^{jk} = 0$ (from eq. 1) (3)

This is true for an arbitrary tensor B^{jk} . Therefore,

$$\bar{A}(\alpha, \beta, \gamma) \frac{\partial \bar{x}^\beta}{\partial x^j} \frac{\partial \bar{x}^\gamma}{\partial x^k} = \frac{\partial \bar{x}^\alpha}{\partial x^i} A(i, j, k)$$

Now multiplying $\frac{\partial x^j}{\partial \bar{x}^p} \frac{\partial x^k}{\partial \bar{x}^\sigma}$ (inner multiplication) on both sides, we obtain

$$\bar{A}(\alpha, \beta, \gamma) \delta_{\beta p} \delta_{\gamma \sigma} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^p} \frac{\partial x^k}{\partial \bar{x}^\sigma} A(i, j, k)$$

$$\bar{A}(\alpha, p, \sigma) = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^p} \frac{\partial x^k}{\partial \bar{x}^\sigma} A(i, j, k) \quad \text{--- (4)}$$

From eq. (4) we see that $A(i, j, k)$ is a tensor of contravariant rank one and covariant rank 2. Now writing eq. (4) in tensor notation

$$\bar{A}^{\alpha}_{p\sigma} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^p} \frac{\partial x^k}{\partial \bar{x}^\sigma} A^{i}_{jk} \quad \text{--- (5)}$$

Conjugate or reciprocal tensors

Let $A_{ij} \rightarrow$ symmetric covariant tensor of rank 2 such that when A_{ij} is expressed as a matrix then $\det(A_{ij}) = |A_{ij}| \neq 0$. Now define

$$B^{ij} = \frac{\text{cofactor of } A_{ij}}{|A_{ij}|}$$

Then B^{ij} is a symmetric contravariant tensor of rank 2 called as the conjugate tensor of A_{ij} and

$$B^{ij} A_{kj} = \delta^i_k$$